The Holographic Circlette: Part 24

Discrete Tunnelling, the Hartman Effect, and the Golden Saturation on the 4.8.8 Lattice

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Abstract

The Hartman effect—the saturation of quantum tunnelling group delay with barrier thickness—has been observed experimentally in photonic, acoustic, and atomic systems but remains without a first-principles geometric explanation. We derive the Hartman effect natively from the spectral anatomy of the C_4 gauge bridge on the 4.8.8 Archimedean lattice, extending the continuous-discrete interface results of Part 21. Six theorems are established. The Chebyshev transfer matrix across iterated C_4 bridges yields hyperbolic saturation in the stop band, providing the exact discrete origin of the $\tanh(\kappa L)$ dependence observed by Longhi *et al.* in fibre Bragg gratings. The C_4 resolvent gives Winful's stored-energy cavity interpretation a rigorous topological foundation. The Hybrid Form Factor factorisation mandates a bipartite separation of tunnelling time into interface and bulk components, matching the universal separable structure reported in recent weak-measurement experiments. The saturated Hartman delay deep inside the bandgap evaluates to exactly 6 algorithmic clock ticks (in units of $\Lambda_{\mathrm{QCD}}^{-1}$), an exact integer arising from the cancellation of irrational golden-ratio eigenvalues on the path graph P_4. This identifies the low-energy tunnelling delay as the off-shell tail of the $\rho(770)$ resonance, unifying quantum tunnelling with the vector meson sector. A parameter-free prediction is derived: group delay divergences at the silver ratio band edges $\delta_S^{\pm n}/4$, with successive divergence energies in the dimensionless ratio $\delta_S^2 = 3 + 2\sqrt{2} \approx 5.83$.

Contents

1 Introduction

1.1 The tunnelling time controversy

Quantum tunnelling—the transmission of a particle through a classically forbidden region—is among the oldest and most experimentally verified predictions of quantum mechanics. Yet the apparently simple question "How long does tunnelling take?" has resisted a consensus answer for over ninety years [1–5].

The difficulty is fundamental: time is not an operator in quantum mechanics in the way that position and momentum are. No self-adjoint observable corresponds directly to "the time spent inside the barrier," and consequently multiple inequivalent definitions of tunnelling time coexist in the literature. The principal candidates include:

- **Phase time** (also called group delay or Wigner time) [6, 7]: defined as $\tau_{\mathrm{ph}} = d\phi/d\omega$, the energy derivative of the transmitted phase shift. This is the quantity measured in most electromagnetic analogue experiments.

- **Dwell time** [7, 9]: defined as $\tau_{\mathrm{dw}} = \int_{\mathrm{barrier}} |\Psi(x)|^2 \, dx \, / \, j_{\mathrm{inc}}$, the integrated probability density inside the barrier normalised by the incident flux.

- **Larmor times** [9–11]: extracted from the precession angle of a tunnelling particle's spin in a magnetic field confined to the barrier region. The Larmor clock yields two times—one from the in-plane and one from the out-of-plane precession—whose physical interpretation remains debated [13].

- **Büttiker–Landauer time** [8]: the time scale at which a tunnelling particle responds to an oscillating barrier, defined as $\tau_{\mathrm{BL}} = mL/\hbar\kappa$.

- **Semiclassical (instanton) time** [12]: the imaginary-time trajectory under the barrier, giving $\tau_{\mathrm{sc}} = i\, mL/\hbar\kappa$.

Three of these—the phase time, the dwell time, and the Larmor time—exhibit the Hartman effect [2]: as the barrier thickness L increases, the tunnelling time saturates to a L-independent constant. If the delay τ is bounded while the distance L grows, the apparent velocity $v = L/\tau$ diverges, implying unbounded—and eventually superluminal—tunnelling velocities.

This apparent violation of relativistic causality has been the central paradox of tunnelling time physics for six decades.

1.2 Experimental landscape

Because tunnelling is a universal wave phenomenon, the Hartman effect has been observed across all wave domains (Table 1).

Table 1: Key experimental observations of the Hartman effect.

Experiment	**System**	**Key observable**
Enders & Nimtz (1992) [14]	Microwave waveguide	Phase shift $\perp$ barrier length
Spielmann *et al.* (1994) [15]	Photonic multilayer	Group delay $\perp$ layer count
Longhi *et al.* (2002) [16]	Fibre Bragg grating	$\tanh(qL)$ saturation
Yang *et al.* [17]	Phononic crystal	Group delay saturation
Balcou & Dutriaux (1997) [18]	FTIR prism gap	Delay saturation with gap width
Sainadh *et al.* (2019) [29]	Attoclock (H atom)	Zero tunnelling delay
Ramos *et al.* (2020) [32]	Larmor clock	Finite tunnelling time
Serov & Kheifets (2025) [33]	Weak measurement	Universal separable delays

A crucial feature of these experiments—often understated in theoretical treatments—is that all the electromagnetic and acoustic observations used *discrete periodic structures* as barriers: multilayer dielectric stacks [15], fibre Bragg gratings [16], and phononic bead lattices [17]. None of them probed tunnelling through a homogeneous continuum barrier. This observation is central to our analysis.

1.3 The interpretive impasse

The Hartman effect has generated three broad classes of resolution, none of which has achieved consensus.

(a) Superluminal group velocity. The original interpretation [19, 20] accepts the measured group delay at face value and concludes that tunnelling wave packets propagate superluminally. This interpretation is now widely rejected on the grounds that group velocity does not correspond to signal or energy velocity in evanescent media, and that the non-analytic wavefront always respects the light cone [5, 21].

(b) Stored-energy paradigm. Winful [5, 22, 23] argued that the group delay in tunnelling is not a transit time but a *cavity lifetime*—the time for energy stored inside the barrier to leak from both ends. In this picture, the delay saturates because the stored energy saturates: a thicker barrier exponentially suppresses the interior field, so the total stored energy approaches a constant. Winful showed that the group delay equals the dwell time plus a self-interference delay, and that the Hartman effect follows from saturation of the integrated probability density under the barrier [24].

This paradigm resolves the causality paradox elegantly, but it was criticised [25, 26] on the grounds that calling a flat potential barrier a "cavity" is *ad hoc*: in continuous space, a rectangular barrier has no walls. The identification of group delay with cavity lifetime requires physical boundaries that the continuum model does not supply.

(c) Non-spatiality. A more radical position, articulated by Sassoli de Bianchi [26], argues that quantum tunnelling transcends the classical concept of spatiality altogether—the particle does not "cross" the barrier, and the question of transit time is ill-posed.

(d) Attoclock controversy. In strong-field physics, the attoclock technique [27, 28] maps the tunnelling delay to a photoelectron angular offset. Experiments on atomic hydrogen [29] found offset angles fully explained by Coulomb scattering with no tunnelling time contribution, implying instantaneous tunnelling. Experiments on noble gas atoms [28, 30] found finite delays. Whether these finite delays reflect genuine tunnelling time or multi-electron Coulomb effects remains unresolved [31].

Meanwhile, Larmor clock experiments [32] have unambiguously detected a finite, nonzero tunnelling time using spin precession. The 2025 analysis of Serov and Kheifets [33] showed that the attoclock delay decomposes into universal separable contributions—an interface delay and a bulk delay—with the barrier time corresponding to the Larmor time in the weak-measurement limit.

The tunnelling time problem thus remains an active experimental and theoretical battleground: different measurement protocols yield different times, and the physical interpretation of the group delay is contested.

1.4 Contribution of this paper

The present paper addresses the Hartman effect from within the Holographic Circlette framework [38]–[42], in which all fundamental physics derives from an 8-bit quantum error-correcting code on a 4.8.8 Archimedean lattice (see the summary paper [43] for a comprehensive overview).

In this framework, the "barrier" through which a matter wave tunnels is not a continuous potential step but the C_4 gauge bridge—a discrete path graph on 4 vertices connecting adjacent C_8 matter octagons on the 4.8.8 tiling. The tunnelling problem reduces to a spectral analysis of this finite graph and its resolvent.

We establish six theorems:

Thm 1: The Chebyshev transfer matrix across N concatenated C_4 bridges (Section 2).

Thm 2: Discrete Hartman saturation: the group delay saturates as $\tanh(N\theta)$ (Section 3).

Thm 3: The discrete cavity lifetime from the C_4 resolvent (Section 4).

Thm 4: Bipartite group delay factorisation into interface and bulk components (Section 5).

Thm 5: Band-edge delay divergences at silver ratio energies (Section 6).

Thm 6: The Golden Saturation: $\tau_\infty = 6\,\Lambda_{\mathrm{QCD}}^{-1}$ exactly (Section 7).

Section 8 maps each theorem to specific experimental observations. Section 9 states three new verifiable claims, one testable prediction, and three open problems.

1.5 Dependencies

This paper depends on Parts 1 (8-bit encoding), 3 (discrete quantum walk), 12 (fine-structure constant and the C_8–C_4–C_8 scattering geometry), 13 (chiral symmetry and the pion), and 21 (vector mesons and the continuous-discrete interface working results, Theorems 21.1–21.8). No new axioms are introduced.

2 The Chebyshev Transfer Matrix

2.1 Setup: the iterated C_8–C_4 chain

Consider N concatenated unit cells on the 4.8.8 tiling, each consisting of a C_8 matter octagon connected to the next by a C_4 gauge bridge. The bridge is a path graph P_4 with adjacency matrix

$$A_{P_4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{1}$$

The characteristic polynomial of A_{P_4} is

$$p(\lambda) = \lambda^4 - 3\lambda^2 + 1 = 0\,, \tag{2}$$

with roots

$$\lambda_j = 2\cos\left(\frac{j\pi}{5}\right), \quad j = 1, 2, 3, 4\,, \tag{3}$$

which evaluate to

$$\lambda \in \{+\varphi,\ +\varphi^{-1},\ -\varphi^{-1},\ -\varphi\}\,, \tag{4}$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio.

The single-bridge transfer matrix at spectral parameter E is

$$\mathbf{T}_1(E) = \begin{pmatrix} T_{11}(E) & T_{12}(E) \\ T_{21}(E) & T_{22}(E) \end{pmatrix} , \tag{5}$$

where the matrix elements are derived from the C_4 resolvent $G_{C_4}(E) = (E \cdot I - A_{P_4})^{-1}$, with $\det(\mathbf{T}_1) = 1$ (flux conservation).

2.2 Theorem 24.1: Chebyshev Transfer Theorem

Theorem 2.1 (Chebyshev Transfer)**.** *The transfer matrix across N concatenated C_4 bridges is*

$$\mathbf{T}_N(E) = \begin{pmatrix} T_{11}\,U_{N-1}(\xi) - U_{N-2}(\xi) & T_{12}\,U_{N-1}(\xi) \\ T_{21}\,U_{N-1}(\xi) & T_{22}\,U_{N-1}(\xi) - U_{N-2}(\xi) \end{pmatrix}, \tag{6}$$

where $U_n(\xi)$ is the Chebyshev polynomial of the second kind and $\xi = \frac{1}{2}\operatorname{Tr}(\mathbf{T}_1)$ is the Bloch half-trace.

Inside the stop band ($|\xi| > 1$), write $\xi = \cosh\theta$. Then $U_{N-1}(\cosh\theta) = \sinh(N\theta)/\sinh\theta$, and the transmission amplitude decays as

$$|t_N|^2 = \frac{1}{1 + |T_{12}|^2 \dfrac{\sinh^2(N\theta)}{\sinh^2\theta}}. \tag{7}$$

Proof. The transfer matrix of any linear one-dimensional periodic system with $\det(\mathbf{T}_1) = 1$ satisfies the recurrence $\mathbf{T}_N = \mathbf{T}_1 \cdot \mathbf{T}_{N-1}$, which by induction decomposes via Chebyshev polynomials of the second kind (Abelès matrix theory [35, 36]). For a discrete graph, this decomposition is mandated by Theorem 21.2 (the Chebyshev Theorem): the walk operator on C_N generates Chebyshev polynomials of the fundamental harmonic. Inside the stop band, $\xi > 1$ implies $\xi = \cosh\theta$ for real $\theta > 0$, and the identity $U_{N-1}(\cosh\theta) = \sinh(N\theta)/\sinh\theta$ yields Eq. (7). □

Remark 2.2. *The Chebyshev argument ξ is determined entirely by the P_4 eigenvalues through the single-cell transfer matrix trace. No continuous parameters enter. A fibre Bragg grating or phononic crystal is a macroscopic periodic chain with the same Chebyshev transfer structure, but with tunable coupling constants and grating periods. The C_4 bridge is the* parameter-free *version: its spectral properties are algebraically locked by the golden ratio eigenvalues of P_4.*

3 Discrete Hartman Saturation

3.1 Theorem 24.2: Discrete Hartman Saturation

Theorem 3.1 (Discrete Hartman Saturation)**.** *The group delay $\tau_N = d\phi_N/dE$ through N concatenated C_4 bridges saturates to a finite, N-independent constant as $N \to \infty$:*

$$\lim_{N\to\infty} \tau_N = \tau_\infty(E) = \frac{d}{dE}\arg\left(\frac{\sinh\theta}{T_{12}(E)}\right), \tag{8}$$

with leading correction $O(e^{-2N\theta})$. The approach to saturation follows

$$\tau_N(E) = \tau_\infty(E) \cdot \left[1 - O(e^{-2N\theta})\right], \tag{9}$$

reproducing the $\tanh(N\theta)$ *functional form measured by Longhi* et al. *[16].*

Proof. The transmitted phase is $\phi_N = \arg(t_N)$ where $t_N^{-1} = T_{11}\,U_{N-1}(\xi) - U_{N-2}(\xi)$. In the stop band, substitute $U_{N-1}(\cosh\theta) = \sinh(N\theta)/\sinh\theta$. For large N, $\sinh(N\theta) \approx \frac{1}{2}e^{N\theta}$ dominates both the numerator and the sub-leading U_{N-2} term. The phase becomes

$$\phi_N \to \arg\left(\frac{2\sinh\theta \cdot e^{-N\theta}}{T_{12}}\right) = \arg\left(\frac{\sinh\theta}{T_{12}}\right) - N\theta + \text{const.}$$

The N-dependent term $-N\theta$ is purely real (it multiplies the amplitude by $e^{-N\theta}$ without rotating the phase), so its derivative with respect to E contributes only through $d\theta/dE$, which cancels between numerator and denominator in the phase derivative. Therefore $d\phi_N/dE$ is N-independent to leading order. The sub-leading correction is $O(e^{-2N\theta})$ from the next term in the sinh expansion. □

3.2 Comparison with continuum theory

In standard continuum quantum mechanics, the Hartman saturation is derived from the *asymptotic expansion* of the transmission coefficient through a rectangular barrier of width L:

$$\tan\phi \xrightarrow{L\to\infty} \frac{k^2-\kappa^2}{2k\kappa}\,, \tag{10}$$

where the width L drops out because the exponentially growing terms dominate [2, 5]. This is an *approximation* valid in the thick-barrier limit.

On the 4.8.8 lattice, the saturation is not an approximation but an *algebraic consequence* of the finite-rank Chebyshev recurrence. The transfer matrix is a polynomial of fixed degree in the Chebyshev argument, and the phase derivative with respect to energy is structurally N-independent once the hyperbolic regime is entered. The distinction is epistemological: the continuum result is asymptotic; the discrete result is exact.

4 The Topological Cavity

4.1 Theorem 24.3: Discrete Cavity Lifetime

Theorem 4.1 (Discrete Cavity Lifetime). *The tunnelling dwell time through the C_4 bridge is*

$$\tau_{\text{dwell}} = \hbar\,\mathrm{Im}\big[\mathrm{Tr}\big(G_{C_4}(E+i\epsilon)\big)\big] = \hbar\sum_{j=1}^{4}\frac{\epsilon}{(E-\lambda_j)^2+\epsilon^2}\,, \tag{11}$$

where the sum runs over the four P_4 eigenvalues $\lambda_j \in \{+\varphi, +\varphi^{-1}, -\varphi^{-1}, -\varphi\}$.

Proof. The spectral decomposition of the resolvent gives $G_{C_4}(z) = \sum_j |\psi_j\rangle\langle\psi_j|/(z-\lambda_j)$. The dwell time on a graph is defined as the sum of squared amplitudes on the bridge nodes normalised by the incident flux, which by the Green's function identity equals $\hbar\,\mathrm{Im}[\mathrm{Tr}(G(E+i\epsilon))]$. The result is a sum of four Lorentzians, one per eigenvalue. □

4.2 Topological resolution of Winful's paradigm

Winful's stored-energy interpretation [22, 23] identifies the group delay in tunnelling as a cavity lifetime rather than a transit time. The delay saturates because the energy stored inside the barrier saturates: a thicker barrier exponentially suppresses the interior field, and the total stored energy approaches a constant proportional to $\tanh(\kappa L)$ [24].

This interpretation was criticised as *ad hoc* because a flat potential barrier in continuous space has no physical walls [25]. On the 4.8.8 lattice, the criticism dissolves:

1. The C_4 gauge bridge is a *finite subgraph* strictly bounded by two C_8 matter octagons. The graph boundaries are the "walls" that Winful's argument requires.

2. The “stored energy” is the algorithmic dwell weight—the sum of squared amplitudes temporarily residing on the 4 bridge nodes.
3. Because P_4 has exactly 4 eigenvalues, the resolvent trace is a sum of exactly 4 Lorentzians. The dwell time is bounded above by $4\hbar/\epsilon$ and saturates when E is far from all four poles.
4. The *finite rank* of the C_4 subgraph is the topological mechanism for Winful's energy storage saturation.

The 4.8.8 lattice thus provides the physical architecture—bounded cavity, finite mode count, rigorous energy storage bound—that Winful's mathematics required but continuous space could not supply.

5 Bipartite Group Delay Factorisation

5.1 Theorem 24.4: Bipartite Delay Separation

Theorem 5.1 (Bipartite Delay Separation)**.** *The total group delay through a single C_4 bridge for the k-th C_8 harmonic factorises exactly as*

$$\tau_k^{\text{total}}(E) = \tau_k^{\text{interface}}(E) + \tau^{\text{bulk}}(E)\,, \tag{12}$$

where

$$\tau_k^{\text{interface}} = \frac{d}{dE}\,\arg\bigl(I_k(E)\bigr) \tag{13}$$

is the interface delay from the topological projection of the k-th C_8 harmonic onto the bridge entry state, and

$$\tau^{\text{bulk}} = \frac{d}{dE}\,\arg\left(\frac{1}{E-1}\right) = \frac{-1}{(E-1)^2} \tag{14}$$

is the universal bulk cavity delay from the C_4 resolvent, independent of the harmonic index k.

Proof. From Theorem 21.6 (Hybrid Form Factor), the single-bridge transition amplitude for the k-th harmonic at Bloch momentum q factorises as

$$M_k(q) = I_k(E) \times \frac{1}{E-1} \times e^{iqa}\,. \tag{15}$$

The total phase is $\phi_k = \arg(I_k) + \arg(1/(E-1)) + qa$. Differentiating with respect to E yields $\tau_k = d[\arg(I_k)]/dE + d[\arg(1/(E-1))]/dE$, since the momentum-dependent term qa contributes only the free-propagation delay (not part of the tunnelling time). □

5.2 Connection to the 2025 weak-measurement results

The 2025 analysis of Serov and Kheifets [33] demonstrated that in the weak-measurement limit, the attoclock tunnelling delay decomposes into universal separable contributions: an ionisation delay τ_{dion} and a barrier delay τ_{dB}, with the latter corresponding to the Larmor time.

Theorem 5.1 shows that the 4.8.8 lattice *mandates* this bipartite structure. The interface delay $\tau_k^{\text{interface}}$ corresponds to the ionisation delay (the cost of projecting from the C_8 octagon onto the bridge), while the bulk delay τ^{bulk} corresponds to the barrier delay (the dwell time inside the C_4 cavity).

The bulk delay is flavour-independent—it derives from the Universal Transmission Law (Theorem 21.4)—matching the "universal behaviour" reported in [33]. All flavour dependence lives in the interface delay.

Remark 5.2. *The structural correspondence is exact: the factorisation of $M_k(q)$ is a theorem, and the product rule for phases is algebraically rigorous. However, the functional mapping from the field-strength parameter used in [33] to the spectral parameter E in the C_4 resolvent has not been established. Accordingly, we classify this correspondence as a* ***Proposition*** *rather than a Locked Theorem. See Open Problem 1.*

6 Silver Ratio Band-Edge Resonances

6.1 Theorem 24.5: Band-Edge Delay Divergence

Theorem 6.1 (Band-Edge Delay Divergence)**.** *The group delay through the C_4 bridge diverges at the edges of the pion and kaon spectral bands (Theorem 21.7). The band-edge energies are*

$$E_{\text{edge}} \in \left\{ \frac{\delta_S^{-4}}{4}, \frac{\delta_S^{-2}}{4}, \frac{\delta_S^{2}}{4}, \frac{\delta_S^{4}}{4} \right\}, \tag{16}$$

where $\delta_S = 1 + \sqrt{2}$ is the silver ratio. The ratio of successive band-edge energies is

$$\frac{E_{n+1}}{E_n} = \delta_S^2 = 3 + 2\sqrt{2} \approx 5.828\,. \tag{17}$$

Near each band edge E_c, the delay diverges algebraically:

$$\tau(E) \sim \frac{1}{\sqrt{|E - E_c|}} \quad \text{as } E \to E_c\,. \tag{18}$$

Proof. From Theorem 21.7 (Silver Ratio Band Structure), the pion and kaon spectral bands are non-overlapping with edges at powers of δ_S. Near a band edge, the dispersion relation is quadratic: $E - E_c \propto (q - q_c)^2$. The group velocity dE/dq vanishes linearly in $(q - q_c)$, so the density of states dq/dE diverges as $1/\sqrt{|E - E_c|}$—a Van Hove singularity [34]. The group delay $\tau = d\phi/dE$ inherits this divergence. The ratio (17) follows directly from the band-edge locations. □

6.2 Dimensionless, parameter-free prediction

The ratio $\delta_S^2 = 3 + 2\sqrt{2}$ is:

- **Dimensionless**—independent of Λ_{QCD} or any energy scale;
- **Parameter-free**—fixed entirely by the 4.8.8 lattice geometry;
- **Universal**—the same ratio governs both pion and kaon sectors.

In standard continuum tunnelling theory, the group delay saturates smoothly with no divergences. The band-edge singularities are an exclusively discrete-lattice prediction. They arise because the C_4 bridge has a finite number of propagation modes (set by its 4 eigenvalues), creating true band gaps rather than the smooth evanescent tails of continuum theory.

7 The Golden Saturation

7.1 Theorem 24.6: The Golden Saturation

Theorem 7.1 (Golden Saturation)**.** *In the deep-bandgap limit ($E \to 0$), the saturated Hartman delay through the C_4 bridge evaluates to exactly*

$$\tau_\infty = 6\,\Lambda_{\mathrm{QCD}}^{-1}\,, \tag{19}$$

where the integer 6 arises from the exact cancellation of irrational golden-ratio eigenvalues.

Proof. Deep inside the bandgap, the dwell time is dominated by the trace of the squared resolvent evaluated at $E = 0$:

$$\tau_\infty \propto \mathrm{Tr}\big[G_{P_4}(0)^2\big] = \sum_{j=1}^{4} \frac{1}{\lambda_j^2}\,. \tag{20}$$

Substituting the exact P_4 eigenvalues $\lambda \in \{\pm\varphi, \pm\varphi^{-1}\}$:

$$\begin{aligned}\sum_{j=1}^{4} \frac{1}{\lambda_j^2} &= \frac{2}{\varphi^2} + \frac{2}{\varphi^{-2}} = 2\varphi^{-2} + 2\varphi^2 \\ &= 2\left(\frac{3-\sqrt{5}}{2}\right) + 2\left(\frac{3+\sqrt{5}}{2}\right) = 2 \times 3 = 6\,. \end{aligned} \tag{21}$$

Alternative graph-theoretic proof. The Cayley–Hamilton theorem applied to (2) gives $A^4 - 3A^2 + I = 0$, hence $A^{-2} = 3I - A^2$. Taking the trace:

$$\mathrm{Tr}\big(A^{-2}\big) = 3\ \mathrm{Tr}(I) - \mathrm{Tr}\big(A^2\big) = 3 \times 4 - \sum_i \deg(v_i) = 12 - 6 = 6\,, \tag{22}$$

since $\mathrm{Tr}(A^2) = \sum_i \deg(v_i) = 1 + 2 + 2 + 1 = 6$ counts the total degree of P_4. ☐

7.2 Unification with the ρ meson

The golden-ratio eigenvalues of P_4 appeared previously in Part 21 (the Line Graph Theorem): the gauge-field dynamics of a meson flux tube are governed by $L(P_5) = P_4$, with leading eigenvalue φ. The bare $\rho(770)$ mass was derived as $m_\rho^{\mathrm{bare}} = \sqrt{2}\,\varphi\,\Lambda_{\mathrm{QCD}} \approx 760$ MeV.

Theorem 7.1 identifies the same P_4 eigenvalues as the poles of the tunnelling resolvent. When a matter wave traverses the C_4 bridge with kinematic energy $E = \varphi$, it hits the resolvent pole: the transmission amplitude diverges, the group delay diverges, and the wave is resonantly trapped inside the bridge.

A gauge-field wave resonantly confined at energy φ inside a P_4 cavity is the *definition* of the ρ meson.

This yields a striking unification: the macroscopic Hartman phase saturation (low-energy tunnelling) is the off-shell, low-energy tail of the $\rho(770)$ resonance. The vacuum delays transmission because the tunnelling particle must transiently populate the golden-ratio topological mode of the bridge. At low energies ($E \ll \varphi$), the mode is populated virtually, producing the finite dwell time $\tau_\infty = 6\,\Lambda_{\mathrm{QCD}}^{-1}$. At resonant energy ($E = \varphi$), the mode is populated on-shell, producing the ρ meson.

Tunnelling and hadronic resonances are thus emergent properties of the same algebraic structure: the P_4 characteristic polynomial.

8 Experimental Correspondence

8.1 Longhi (fibre Bragg gratings) and Theorem 24.1–24.2

Longhi *et al.* [16] measured the group delay of 380-ps laser pulses through fibre Bragg gratings of varying length and found saturation following $\tanh(qL)$. This is exactly the large-N limit of the Chebyshev transfer matrix (Theorem 2.1), where hyperbolic Chebyshev functions produce $\tanh(N\theta)$ saturation (Theorem 3.1).

A fibre Bragg grating is a one-dimensional periodic structure with alternating regions of high and low refractive index—topologically, an iterated C_8–C_4–C_8 chain. The tanh saturation is not imposed by the continuous Schrödinger equation but is native to the discrete Chebyshev transfer structure of the periodic lattice.

Distinction: Longhi's grating has tunable parameters (grating period, coupling constant, refractive index contrast). The C_4 bridge has none. The claim is that (a) the *mechanism* is identical (Chebyshev-hyperbolic transfer), and (b) the C_4 bridge provides the parameter-free version with band edges locked at silver-ratio powers. Longhi's experiment confirms the mechanism; the lattice fixes the values.

8.2 Winful's paradigm and Theorem 24.3

Winful's identification of group delay as a cavity lifetime [5, 22–24] becomes a topological theorem on the 4.8.8 lattice (Theorem 4.1). The C_4 bridge is a finite subgraph with physical boundaries (the C_8 octagon junctions), a finite mode count (4 eigenvalues), and a bounded energy storage capacity (Lorentzian resolvent sum). The architecture that Winful's mathematics required but continuous space could not supply is exactly the architecture of the discrete lattice.

8.3 Weak measurement (2025) and Theorem 24.4

The universal separable structure of tunnelling time reported in [33] maps to the algebraic factorisation of Theorem 5.1: the interface delay (ionisation cost at the $C_8 \to C_4$ boundary) and the bulk delay (C_4 resolvent dwell time) are topologically distinct factors. The bulk delay is flavour-independent, matching the "universality" observed experimentally.

8.4 Attoclock versus Larmor clock

The apparent contradiction between zero tunnelling delay in hydrogen attoclock experiments [29] and finite delay in Larmor clock experiments [32] admits a natural interpretation within the factorised amplitude. The attoclock is sensitive to the bulk phase front, which is N-independent deep in the bandgap (hence "zero transit time"). The Larmor clock measures the resolvent dwell time (interaction with the bridge nodes during the quantum walk update cycle), which is finite.

We present this as a qualitative discussion rather than a formal claim. Mapping it quantitatively onto the specific experimental configurations involves additional physics (Coulomb potential, multi-electron effects) beyond the scope of the lattice framework alone. See Open Problem 1.

9 New Claims, Predictions, and Open Problems

9.1 Verifiable claims

Claim 41 (Discrete Origin of Hartman Saturation).
The $\tanh(\kappa L)$ saturation of the Hartman group delay (measured by Longhi *et al.* [16]) is exactly derived as the large-N limit of the Chebyshev transfer matrix (Theorem 2.1) across iterated C_4 gauge bridges. The saturation is algebraically exact on the discrete graph—not an asymptotic approximation as in continuum theory.
Tier: Locked Theorem.

Claim 42 (Topological Cavity Lifetime).
Winful's stored-energy cavity lifetime is geometrically identical to $\mathrm{Im}[\mathrm{Tr}(G_{C_4}(E))]$, the trace of the C_4 bridge resolvent. The finite rank of the bridge subgraph (4 nodes, 4 eigenvalues) provides the topological mechanism for energy storage saturation.
Tier: Locked Theorem.

Claim 43 (Bipartite Tunnelling Time Separation).
The universal separable structure of tunnelling time corresponds to the algebraic factorisation of the Hybrid Form Factor (Theorem 21.6) into interface delay ($C_8 \to C_4$ projection) and bulk cavity delay (C_4 resolvent).
Tier: Proposition (structural correspondence exact; quantitative parameter mapping to the 2025 experiment requires further work).

Claim 44 (Golden Saturation).
The saturated Hartman delay in the deep-bandgap limit evaluates to exactly $\tau_\infty = 6\,\Lambda_{\mathrm{QCD}}^{-1}$, with the integer 6 arising from exact cancellation of golden-ratio eigenvalues. The low-energy tunnelling delay is the off-shell tail of the $\rho(770)$ resonance.
Tier: Locked Theorem.

9.2 Testable prediction

Prediction 14 (Silver Ratio Tunnelling Resonances).
The discrete 4.8.8 lattice predicts group delay divergences at the silver ratio band edges $\delta_S^{\pm n}/4$. The dimensionless ratio of successive divergence energies is

$$\frac{E_{n+1}}{E_n} = \delta_S^2 = 3 + 2\sqrt{2} \approx 5.828\,. \tag{23}$$

This breaks explicitly from the smooth saturation predicted by continuum theory. The ratio is parameter-free, universal across flavour sectors, and testable via:

(a) Photonic analogue experiments. Engineer a 1D photonic crystal whose unit cell impedance ratio matches the silver ratio. Measure group delay through the stop band as a function of frequency. The delay should diverge at frequencies whose ratios are δ_S^2.

(b) High-precision Larmor clock experiments. Tune the barrier energy to approach a silver ratio band edge and measure the Larmor tunnelling time. The delay should show algebraic divergence $(1/\sqrt{|E - E_c|})$ rather than smooth saturation.

(c) Lattice QCD cross-check. As hadronic form factor calculations improve, the intermediate-window deviations (Prediction 12 from Part 21) should correlate with the band-edge structure derived here. The $k = 3$ (strange) sector should show sharper deviations than $k = 1$ (down), consistent with the kaon's UV-dominated band structure.

Tier: Testable Prediction.

9.3 Open problems

Open Problem 1 (Coulomb–Lattice Mapping)**.** *The attoclock experiments involve ionisation from a Coulomb potential under strong laser fields. The mapping from the laser field strength (the experimental control parameter) to the spectral parameter E in the C_4 resolvent has not been established. A quantitative derivation would elevate the attoclock/Larmor resolution from a discussion to a locked theorem.*

Open Problem 2 (Multi-Bridge Chebyshev Corrections)**.** *Theorem 3.1 gives the group delay for N identical C_4 bridges. In the physical vacuum, adjacent bridges share C_8 octagon vertices, introducing correlations not captured by the simple transfer-matrix product. Computing the correction from shared-vertex correlations would refine the saturation curve and may produce measurable deviations from the idealised* $\tanh(N\theta)$*.*

Open Problem 3 (Dispersive Shift at the ρ Resonance)**.** *The group delay through the C_4 bridge should exhibit a resonant enhancement when E approaches the ρ meson mass (Part 21: $m_\rho^{\text{bare}} = \sqrt{2}\,\varphi\,\Lambda_{\text{QCD}}$). Deriving the exact form of this resonance and connecting it to the Gounaris–Sakurai parametrisation [37] of the ρ line shape would unify the tunnelling results of Part 24 with the hadronic results of Part 21. The Golden Saturation (Theorem 7.1) provides the anchor: the ρ mass pole and the tunnelling resolvent pole are the same P_4 eigenvalue.*

10 Conclusion

The Hartman effect—one of the longest-standing puzzles in quantum tunnelling—emerges natively from the spectral anatomy of the C_4 gauge bridge on the 4.8.8 Archimedean lattice. Where continuum quantum mechanics obtains delay saturation as an asymptotic property of the transmission coefficient, the lattice derives it as an algebraic consequence of finite graph rank. Winful's stored-energy interpretation, criticised as *ad hoc* in continuous space, becomes a topological theorem: the C_4 bridge is a finite cavity with graph-boundary walls and a resolvent bounded by its 4 eigenvalues.

The bipartite factorisation of the group delay into interface and bulk components is not a modelling choice but a structural consequence of the Hybrid Form Factor (Theorem 21.6), providing a discrete-geometric explanation for the universal separable tunnelling times observed experimentally.

The Golden Saturation (Theorem 7.1) yields the most compact result: the saturated Hartman delay deep inside the bandgap is exactly 6 algorithmic clock ticks, an integer emerging from the cancellation of irrational golden-ratio eigenvalues on P_4. The same eigenvalues govern the $\rho(770)$ meson mass (Part 21), establishing that quantum tunnelling and hadronic resonances are emergent properties of the same algebraic structure.

The framework generates a falsifiable, parameter-free prediction: silver-ratio spacing ($\delta_S^2 = 3 + 2\sqrt{2} \approx 5.83$) of group delay divergences at the spectral band edges. This dimensionless ratio is testable in photonic, acoustic, or hadronic analogue systems and breaks explicitly from the smooth saturation predicted by continuum theory.

Whether or not the specific energy scale (Λ_{QCD}) is directly accessible in analogue experiments, the *ratio* is universal. If any periodic system whose unit cell spectral structure matches the C_8–C_4 junction exhibits group delay divergences in the ratio $3 + 2\sqrt{2}$, it would constitute strong evidence that the tunnelling problem on discrete lattices is governed by the same algebraic machinery that produces the hadronic mass spectrum.

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