Pauli Antisymmetry from $\mathbb{F}_2$ XOR-Closure: A Discrete-Geometric Derivation of Composite Baryon Statistics

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We show that the antisymmetry of composite baryon wavefunctions under exchange—the empirical content of Pauli's exclusion principle for hadrons—can be derived from the $\mathbb{F}_2$ XOR-closure structure of a discrete-geometric inner code, without invoking continuum Lorentz invariance, microcausality, or the spin-statistics theorem. Working on an 8-qubit internal Hilbert space $\mathcal{Q}_3$ projected onto a 48-codeword physical subspace, we identify three quark colours that XOR-close to the identity element. We show that the resulting global stabiliser Q promotes baryon creation operators to defect-creating operators with a definite anticommutation algebra. Two consequences follow: (i) baryon parity is a $\mathbb{Z}_2$ conserved charge; (ii) the exchange of two baryons produces a phase of -1 from the $\{X, Z\} = 0$ algebra of the underlying Pauli operators on the inner code, provided a consistent path-ordering convention is established on the macroscopic lattice. The same construction, restricted to the strong-interaction channel V_{strong}, gives a tensor decomposition $I_{12} \otimes A_{K_3}$ on the 36-dimensional quark sector, where A_{K_3} is the complete-graph-on-three-vertices adjacency matrix. The induced kinetic operator has eigenvalues $\pm i\, 2/\sqrt{3}$ on colour-singlet states and $\pm i/\sqrt{3}$ on colour-doublet states, energetically favouring colour-singlet bound states by a factor of four. The algebraic claims are verified by exact rational identities in numerical diagonalisation on the projected codeword space. The construction provides a discrete-geometric origin for hadronic Pauli antisymmetry parallel to, but mechanistically independent of, the spin-statistics theorem.

I. INTRODUCTION

Pauli's exclusion principle asserts that the wavefunction of a system of identical fermions is antisymmetric under particle exchange:

$$\psi(x_1, x_2) = -\psi(x_2, x_1). \tag{1}$$

In the framework of relativistic quantum field theory, this is derived rather than postulated. The spin-statistics theorem [1–3] establishes that any local quantum field theory satisfying Lorentz invariance, microcausality, positive energy, and a unique vacuum must assign antisymmetric (anticommuting) statistics to half-integer-spin fields and symmetric statistics to integer-spin fields. The -1 phase under fermion exchange is a consequence of the continuous structure of spacetime together with the local algebraic structure of the field operators.

A natural question is whether the same antisymmetry can arise from a structurally different mechanism—specifically, from algebraic structure intrinsic to a discrete-geometric model that does not assume continuum Lorentz invariance from the outset. We address this question affirmatively for the case of composite baryons in a discrete-geometric setting: a macroscopic spatial lattice equipped with a linear constraint structure over $\mathbb{F}_2$ (the field of two elements).

The construction proceeds as follows. We work on an 8-qubit internal Hilbert space $\mathcal{Q}_3$ hosted at each lattice site, projected onto a 48-dimensional physical subspace by a set of linear $\mathbb{F}_2$ constraints [4]. The constraints distinguish two species of physical states: a 12-dimensional "lepton" sector and a 36-dimensional "quark" sector. The quark sector decomposes further into three colours related by an $\mathbb{F}_2$ XOR-closure relation: the three colour codewords XOR to the zero element, so any two colours determine the third up to closure.

This XOR-closure structure promotes the global parity L_Q (which counts the parity of quark content) to a $\mathbb{Z}_2$ gauge charge with a corresponding stabiliser operator Q. The composite baryon creation operator anticommutes with Q as a defect-creating operator. Under a consistent path-ordering convention, two such operators applied at distinct sites have an exchange algebra that produces the -1 phase characteristic of fermion exchange.

The derivation does not require continuum Lorentz invariance, microcausality, or the spin-statistics theorem. It produces the -1 phase as a consequence of the $\{X, Z\} = 0$ anticommutation algebra of Pauli operators on the inner code, mediated by the $\mathbb{F}_2$ XOR-closure constraint and an algebraic path convention.

We specify the bounds of this result. The derivation establishes Pauli antisymmetry for *composite baryons* only. It does not address the statistics of elementary leptons (which in this framework correspond to localised excitations of a single bit). It does not establish that the framework realises a full Standard Model. Furthermore, it does not establish topological order: the vacuum of the framework is a trivial classical product state, and the framework does not exhibit the 8-fold ground-state degeneracy on the 3-torus that would characterise a deconfined $\mathbb{Z}_2$ topological phase [5, 6]. What it establishes is an alternative discrete-geometric derivation of composite-baryon Pauli antisymmetry, mechanistically independent of the

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spin-statistics theorem.

II. GEOMETRIC SETUP

A. The Internal Hilbert Space

Following previous development of the framework [7], at each site r of a macroscopic simple cubic spatial lattice $\mathbb{Z}^3$, we attach an 8-qubit internal Hilbert space

$$\mathcal{Q}_3 = (\mathbb{C}^2)^{\otimes 8} \cong \mathbb{C}^{256}. \tag{2}$$

We label the eight qubits by indices $f \in \{0, 1, \ldots, 7\}$, which correspond geometrically to the eight faces of an internal oblate square bipyramid. The face-adjacency graph of this internal geometry is the 3-cube Q_3. A basis state of $\mathcal{Q}_3$ is specified by an 8-bit string $c = (c_0, c_1, \ldots, c_7) \in \mathbb{F}_2^8$.

We impose the following $\mathbb{F}_2$ linear constraints to project to the physical subspace:

- **(C1) Charge consistency:** $c_0 \cdot c_1 \neq 1$ (equivalently, $c_0 \wedge c_1 = 0$).
- **(C2) Chirality consistency:** $c_6 = c_7$.
- **(C3) Colour-parity coupling:** If $c_2 = 0$, then $c_3 = c_4 = 0$; if $c_2 = 1$, then $(c_3, c_4) \neq (0, 0)$.

Define the lepton charge $L_Q := c_2$. Constraint (C3) couples L_Q to the bits (c_3, c_4): states with $L_Q = 0$ have $(c_3, c_4) = (0, 0)$; states with $L_Q = 1$ have $(c_3, c_4) \in \{(0, 1), (1, 0), (1, 1)\}$. We refer to (c_3, c_4) as the colour bits of a state with $L_Q = 1$.

Direct enumeration yields the physical state distribution described in Table I. The total physical subspace $\mathcal{V} \subset \mathcal{Q}_3$ has dimension $12 + 36 = 48$.

B. The Colour XOR-Closure

The three colour assignments $(c_3, c_4) \in \{(0, 1), (1, 0), (1, 1)\}$ form a complete set, as their componentwise XOR equals the trivial element:

$$(0, 1) \oplus (1, 0) \oplus (1, 1) = (0, 0). \tag{3}$$

We denote the three colours as $R = (0, 1)$, $G = (1, 0)$, $B = (1, 1)$. The closure relation $R \oplus G \oplus B = 0$ is the structural origin of colour neutrality and of the baryon parity $\mathbb{Z}_2$ gauge structure.

TABLE I. Physical subspace sectors imposed by $\mathbb{F}_2$ constraints on the $\mathcal{Q}_3$ inner code.

Sector	**Constraint**	**Free bits**	**Count**
Leptons ($L_Q = 0$)	$c_3 = c_4 = 0$	c_0, c_1, c_5, c_6, c_7 with (C1), (C2)	$3 \times 2 \times 2$ $= 12$
Quarks ($L_Q = 1$)	$(c_3, c_4) \neq (0, 0)$	c_0, c_1, c_5, c_6, c_7 with (C1), (C2)	12×3 $= 36$

Define the colour Hamming weight of a multi-particle state as the componentwise XOR of all colour-bit pairs (c_3, c_4) summed (mod 2) across particles. The XOR-closure implies that a configuration of three quarks, one each in colours R, G, B, has total colour Hamming weight $(0, 0)$. This is a discrete-geometric analogue of the empirical observation that physical baryons are $SU(3)$ colour singlets.

The baryon parity L_Q^{tot} of a multi-particle state is the sum (mod 2) of the individual L_Q values. For a single quark, $L_Q = 1$; for a three-quark baryon, $L_Q^{\text{tot}} = 1 \oplus 1 \oplus 1 = 1 \pmod 2$. Thus, a three-quark baryon has the same L_Q parity as a single quark; it lies at Hamming distance 1 in the L_Q sector from the lepton vacuum.

III. THE $\mathbb{Z}_2$ GAUGE STRUCTURE

A. The Closure Stabiliser

The $\mathbb{F}_2$ closure structure defines a $\mathbb{Z}_2$ symmetry of the physical subspace. We make this explicit by constructing a global stabiliser operator Q that implements baryon parity conservation.

Let $Z_{L_Q}^{(r)}$ denote the Pauli Z operator acting on the L_Q bit (bit c_2) of site r. Define

$$Q := \prod_{r \in \Lambda} Z_{L_Q}^{(r)} \tag{4}$$

where the product runs over all macroscopic lattice sites Λ. This operator measures the total baryon parity of the global configuration:

$$Q \, |\psi\rangle = (-1)^{\sum_r L_Q(r)} \, |\psi\rangle \, . \tag{5}$$

Configurations with even total L_Q are in the $+1$ eigenspace of Q; configurations with odd total L_Q are in the -1 eigenspace. The two eigenspaces are mutually orthogonal, and the dynamics preserve Q provided all hopping and interaction terms commute with Q. This condition holds for the strong, electromagnetic, and weak channels (each term flips L_Q on an even number of sites or none at all).

B. Baryon Creation as $\mathbb{F}_2$ Defect Creation

Consider a local operator F_r that flips L_Q at site r, acting as an X-type operator on bit c_2. This operator anticommutes with the local $Z_{L_Q}^{(r)}$ factor of Q:

$$X_{L_Q}^{(r)} Z_{L_Q}^{(r)} = -Z_{L_Q}^{(r)} X_{L_Q}^{(r)}. \tag{6}$$

Since all other factors of Q commute with F_r, we have $\{F_r, Q\} = 0$.

Therefore, F_r is a defect-creating operator with respect to Q: acting on a $+1$-eigenstate, it produces a -1-eigenstate. Geometrically, F_r creates (or annihilates) a single unit of baryon parity at site r.

For F_r to be a physical creation operator, it must produce states in the valid physical subspace $\mathcal{V}$. Constraint (C3) requires that flipping L_Q from 0 to 1 must turn on at least one colour bit. The smallest such operator has the form

$$F_r^c = X_{L_Q}^{(r)} \otimes P_r^c, \tag{7}$$

where P_r^c is a projector that ensures the colour bits take the value of colour $c \in \{R, G, B\}$. We interpret the multi-quark structure of a baryon in a second-quantised Fock space built over the single-site physical subspace $\mathcal{V}$: each F_r^c creates a quark mode of colour c at site r, and a baryon at site r is a three-particle Fock excitation with one quark in each colour mode. The composite operator that creates a colour-singlet baryon is then

$$\mathcal{B}_r := F_r^R F_r^G F_r^B. \tag{8}$$

By the $\mathbb{F}_2$ closure $R \oplus G \oplus B = 0$, the composite $\mathcal{B}_r$ is colour-neutral. By the arithmetic $1 \oplus 1 \oplus 1 = 1 \pmod 2$, $\mathcal{B}_r$ flips total L_Q by one. A composite baryon is mathematically equivalent to a single $\mathbb{F}_2$ defect under Q.

C. Wilson Strings and the Endpoint Condition

In a $\mathbb{Z}_2$ gauge theory, single parity defects must be created in pairs connected by a Wilson string of Z-operators. Define a Wilson Z-string along a path γ connecting sites r_1 and r_2 as

$$W_\gamma := \prod_{r \in \gamma} Z_{L_Q}^{(r)}. \tag{9}$$

A composite operator $\mathcal{B}_{r_1} W_\gamma \mathcal{B}_{r_2}$ creates a baryon at r_1 and an anti-baryon at r_2 joined by the Z-string W_γ. The total operator commutes with Q. This constraint mirrors the creation of anyonic excitations in topological string-net codes [5], where charge-carrying operators are string-like.

IV. STRONG-CHANNEL STRUCTURE AND COLOUR-SINGLET PREFERENCE

A. The Strong Interaction Operator

We define the strong-channel matrix element V_{strong} on the 256-dimensional internal space by $\langle c' | V_{\text{strong}} | c \rangle = g_s$ whenever both c and c' satisfy $L_Q = 1$ and the bit-difference between them is contained within $\{c_3, c_4\}$. V_{strong} permutes the three colours of a quark at fixed inner-code values. Projecting to the physical subspace $\mathcal{V}$, the 12-dimensional lepton subspace is annihilated, and the operator possesses support solely on the 36-dimensional quark sector $\mathcal{V}_q$.

B. Tensor Decomposition on the Quark Sector

The 36-dimensional quark sector factorises as $\mathcal{V}_q \cong \mathbb{C}^{12} \otimes \mathbb{C}^3$, where $\mathbb{C}^{12}$ is the inner-code space and $\mathbb{C}^3$ is the colour space. The matrix element specification requires

$$V_{\text{strong}}|_{\mathcal{V}_q} = g_s I_{12} \otimes A_{K_3}, \tag{10}$$

where A_{K_3} is the adjacency matrix of the complete graph K_3 on three vertices.

C. Spectral Structure of the Kinetic Operator

We define the strong-channel hopping operator

$$T^{\text{strong}} := -\frac{i}{\sqrt{3}} V_{\text{strong}}. \tag{11}$$

On the quark sector, squaring the operator gives

$$(T^{\text{strong}})^2|_{\mathcal{V}_q} = -\frac{g_s^2}{3} I_{12} \otimes A_{K_3}^2. \tag{12}$$

The complete graph adjacency matrix satisfies $A_{K_3}^2 = I_3 + J_3$, where J_3 is the 3×3 all-ones matrix. The coupling g_s sets the dimensionful kinetic scale; we set $g_s = 1$ in what follows to display the dimensionless spectrum. The diagonal of $(T^{\text{strong}})^2$ on $\mathcal{V}_q$ evaluates to $-2/3$, and off-diagonal entries within each colour block are $-1/3$.

The eigenvalues of A_{K_3} are $\{2, -1, -1\}$ [8]. The eigenvalue 2 corresponds to the symmetric colour-singlet mode. Squaring and applying the prefactor, the $(T^{\text{strong}})^2$ eigenvalues on $\mathcal{V}_q$ are:

$$-\frac{4}{3} \quad \text{(singlets, 12-fold)}, \tag{13}$$

$$-\frac{1}{3} \quad \text{(doublets, 24-fold)}. \tag{14}$$

In the tight-binding Hamiltonian, the squared kinetic energy contribution for the colour-singlet state is a factor of four greater than the doublet contribution. This ratio is parameter-free: g_s sets the overall energy scale but does not affect the relative weight of singlet versus doublet. The discrete geometry selects colour-neutral bound states without parameter-tuning.

V. DERIVATION OF THE -1 BARYON EXCHANGE PHASE

A. Statement of the Exchange Relation

Let $\mathcal{B}_{r_1}$ and $\mathcal{B}_{r_2}$ be composite baryon creation operators at distinct macroscopic sites r_1, r_2. The operator-algebra content of Pauli antisymmetry requires the exchange relation

$$\widetilde{\mathcal{B}}_{r_1} \widetilde{\mathcal{B}}_{r_2} = -\widetilde{\mathcal{B}}_{r_2} \widetilde{\mathcal{B}}_{r_1} \tag{15}$$

when the operators are dressed with Wilson Z-strings adhering to a consistent spatial path convention.

B. Algebraic Derivation via Path-Ordered String Operators

We work in the algebra generated by the Pauli operators $\{X_{L_Q}^{(r)}, Z_{L_Q}^{(r)}\}_{r\in\Lambda}$ on the macroscopic $\mathbb{Z}^3$ lattice Λ. The defining commutation relations are

$$[X_{L_Q}^{(r)}, X_{L_Q}^{(s)}] = 0, \quad [Z_{L_Q}^{(r)}, Z_{L_Q}^{(s)}] = 0, \tag{16}$$

$$[X_{L_Q}^{(r)}, Z_{L_Q}^{(s)}] = 0 \quad (r \neq s), \tag{17}$$

$$\{X_{L_Q}^{(r)}, Z_{L_Q}^{(r)}\} = 0. \tag{18}$$

The colour-bit operators acting on (c_3, c_4) within each projector P_r^c commute with all L_Q-bit operators and play no role in the exchange algebra; we suppress them in what follows. As established in Sec. III.B, the composite baryon operator $\mathcal{B}_r = F_r^R F_r^G F_r^B$ contains three factors of $X_{L_Q}^{(r)}$, which by $(X_{L_Q}^{(r)})^2 = I$ combine to a single $X_{L_Q}^{(r)}$ acting on the L_Q bit at site r. Modulo the colour projectors, we may therefore identify

$$\mathcal{B}_r \sim X_{L_Q}^{(r)} \tag{19}$$

for the purpose of the exchange computation.

a. Path convention. Impose a total ordering $\prec$ on Λ by lexicographic comparison of integer coordinates (z, y, x): that is, $r \prec r'$ iff $(z_r, y_r, x_r) < (z_{r'}, y_{r'}, x_{r'})$ in lexicographic order. For each $r \in \Lambda$, define the path

$$\gamma_r := \{r' \in \Lambda : r' \prec r\} \tag{20}$$

and the associated Wilson string

$$W_{\gamma_r} := \prod_{r' \in \gamma_r} Z_{L_Q}^{(r')}. \tag{21}$$

By Eq. (16) the Z-operator factors mutually commute, so the product is independent of the order in which the factors are written. The dressed baryon creation operator is

$$\widetilde{\mathcal{B}}_r := X_{L_Q}^{(r)}\, W_{\gamma_r} = X_{L_Q}^{(r)} \prod_{r' \prec r} Z_{L_Q}^{(r')}. \tag{22}$$

b. Exchange computation. Consider two distinct sites $r_1, r_2 \in \Lambda$. Without loss of generality, assume $r_1 \prec r_2$ (the opposite case follows by relabelling). The product of dressed operators is

$$\widetilde{\mathcal{B}}_{r_1} \widetilde{\mathcal{B}}_{r_2} = X_{L_Q}^{(r_1)} W_{\gamma_{r_1}} \cdot X_{L_Q}^{(r_2)} W_{\gamma_{r_2}}. \tag{23}$$

To bring this into the form $\widetilde{\mathcal{B}}_{r_2} \widetilde{\mathcal{B}}_{r_1}$ we proceed in two steps.

Step 1: Move $X_{L_Q}^{(r_2)}$ leftward past $W_{\gamma_{r_1}}$. Each factor $Z_{L_Q}^{(r')}$ in $W_{\gamma_{r_1}}$ has $r' \prec r_1 \prec r_2$, hence $r' \neq r_2$. By Eq. (17), every such factor commutes with $X_{L_Q}^{(r_2)}$, so this passage is sign-free:

$$\widetilde{\mathcal{B}}_{r_1} \widetilde{\mathcal{B}}_{r_2} = X_{L_Q}^{(r_1)} X_{L_Q}^{(r_2)} W_{\gamma_{r_1}} W_{\gamma_{r_2}}. \tag{24}$$

Step 2: Move $X_{L_Q}^{(r_1)}$ rightward past $X_{L_Q}^{(r_2)}$ and through $W_{\gamma_{r_2}}$. The first passage (past $X_{L_Q}^{(r_2)}$) is trivial by Eq. (16). The string γ_{r_2} contains r_1 because $r_1 \prec r_2$, so among the factors of $W_{\gamma_{r_2}} = \prod_{r'' \prec r_2} Z_{L_Q}^{(r'')}$ there is exactly one factor $Z_{L_Q}^{(r_1)}$. All other factors satisfy $r'' \neq r_1$ and commute with $X_{L_Q}^{(r_1)}$ by Eq. (17). The single factor $Z_{L_Q}^{(r_1)}$ anticommutes with $X_{L_Q}^{(r_1)}$ by Eq. (18), yielding one factor of -1:

$$X_{L_Q}^{(r_1)} W_{\gamma_{r_2}} = -W_{\gamma_{r_2}} X_{L_Q}^{(r_1)}. \tag{25}$$

Conversely, $W_{\gamma_{r_1}}$ does *not* contain r_2 since $r_2 \not\prec r_1$, so $X_{L_Q}^{(r_2)}$ commutes with $W_{\gamma_{r_1}}$ entirely and no further sign is generated. Reassembling the operators:

$$\widetilde{\mathcal{B}}_{r_1} \widetilde{\mathcal{B}}_{r_2} = -\, X_{L_Q}^{(r_2)} W_{\gamma_{r_2}} \cdot X_{L_Q}^{(r_1)} W_{\gamma_{r_1}} = -\widetilde{\mathcal{B}}_{r_2} \widetilde{\mathcal{B}}_{r_1}. \tag{26}$$

c. General form. For an arbitrary path assignment $r \mapsto \gamma_r$, the analogous computation yields

$$\widetilde{\mathcal{B}}_{r_1} \widetilde{\mathcal{B}}_{r_2} = (-1)^{n_{12}}\, \widetilde{\mathcal{B}}_{r_2} \widetilde{\mathcal{B}}_{r_1}, \tag{27}$$

where the intersection parity is

$$n_{12} = \big([r_1 \in \gamma_{r_2}] + [r_2 \in \gamma_{r_1}]\big) \bmod 2. \tag{28}$$

For the lexicographic convention, the trichotomy property of a total ordering ensures that exactly one of $r_1 \prec r_2$ or $r_2 \prec r_1$ holds for any distinct pair, giving $n_{12} = 1$ universally and reproducing Eq. (26).

d. Gauge nature of the path convention. The lexicographic ordering is a gauge choice analogous to the site-numbering in the one-dimensional Jordan-Wigner transformation [9, 10]. Alternative total orderings on Λ yield different intermediate operator expressions $\widetilde{\mathcal{B}}_r$ but produce the same physical antisymmetry on two-baryon states: any total ordering gives $n_{12} = 1$ for distinct pairs by the trichotomy property. The framework therefore exhibits canonical fermion antisymmetry independent of the convention, with the convention required only to write down explicit operator expressions in a chosen basis. The dressed operators $\widetilde{\mathcal{B}}_r$ do *not* respect $\mathbb{Z}^3$ rotational symmetry; the rotational invariance of physical observables on antisymmetric two-baryon states is recovered after the antisymmetry property itself is established.

C. The Connection to Pauli's Principle

Restating this result in the language of wavefunctions: let $|r_1, r_2\rangle$ denote a two-baryon state defined over the product vacuum $|\mathrm{vac}\rangle$:

$$|r_1, r_2\rangle := \widetilde{\mathcal{B}}_{r_1} \widetilde{\mathcal{B}}_{r_2}\, |\mathrm{vac}\rangle\,. \tag{29}$$

By the anticommuting exchange relation Eq. (26), swapping the operators yields

$$|r_2, r_1\rangle = \widetilde{\mathcal{B}}_{r_2}\widetilde{\mathcal{B}}_{r_1}|\mathrm{vac}\rangle$$
$$= -\widetilde{\mathcal{B}}_{r_1}\widetilde{\mathcal{B}}_{r_2}|\mathrm{vac}\rangle = -|r_1, r_2\rangle . \qquad (30)$$

The two-baryon wavefunction is antisymmetric under exchange. This derivation extracts the -1 phase directly from the discrete $\{X, Z\} = 0$ inner-code algebra and the path-ordered string geometry, bypassing the continuous Lorentz representation theory underpinning the spin-statistics theorem.

VI. DISCUSSION

A. What This Derivation Establishes

The antisymmetry of the two-baryon wavefunction under exchange follows from the $\mathbb{F}_2$ closure algebra combined with the local Pauli algebra. The mechanism generalises the Jordan-Wigner framework to three dimensions via an algebraic path ordering. The closure structure simultaneously provides an origin for colour neutrality and the factor-4 energetic preference for colour singlets.

B. Scope and Limitations

We define the bounds of these results:

1. **Lepton statistics are unaddressed.** Elementary leptons are single-bit excitations lacking the composite 3-colour closure structure. Their statistics demand a separate derivation.

2. **The vacuum is not a topological condensate.** The lowest-energy state is a classical product state. As verified in Appendix A, Wilson loop operators act trivially on this vacuum ($\langle\mathrm{vac}| W_X |\mathrm{vac}\rangle = 1.0$). We do not observe the ground-state degeneracy on the 3-torus characteristic of a deconfined $\mathbb{Z}_2$ topological phase.

3. **Independence from Spin-Statistics.** The framework establishes that the empirical -1 phase possesses an alternative discrete origin, not that the standard derivation is superseded.

C. Relation to Other Discrete-Geometric Approaches

The construction shares structural relations with several discrete frameworks. The colour XOR-closure echoes the colour-flux structure of Bombín's 3D topological colour codes [6] and the Walker-Wang model [11], which both use $\mathbb{F}_2$-like closures to generate non-trivial gauge behaviour in three dimensions.

The path-ordered Wilson-string dressing of baryon operators is analogous to 3D algebraic extensions of the Jordan-Wigner transformation [9, 10]. The factor-4 energetic preference for colour-neutral states parallels the dimer-singlet preference in Anderson's resonating valence bond (RVB) states [12].

What is distinctive about the present construction is the explicit derivation of Pauli antisymmetry for composite particles from the $\mathbb{F}_2$ inner-code closure alone, achieved from a trivial product vacuum without invoking topological order.

D. Implications for Future Work

The empirical content of Pauli's principle for hadrons does not demand continuum spacetime or relativistic field theory as a prerequisite.

An open question involves the electromagnetic and weak channels. While we have restricted the analytical decomposition to the strong-channel base, numerical evaluations of the full hopping operator $T = -i(V_{\mathrm{em}} + V_{\mathrm{weak}} + V_{\mathrm{strong}})/\sqrt{3}$ reveal residual anticommutator norms $\|\{T_i, T_j\}\| \sim 2/3$. These residuals encode the action of V_{em} (diagonal, charge-related) and V_{weak} (off-diagonal, chirality-mixing) on top of the strong-channel Clifford-like base. Their structural origin in the chirality and charge sectors of the inner code, and their possible interpretation as mass and chiral-coupling corrections to the Standard Model's gauge structure, is the subject of forthcoming work.

VII. CONCLUSION

The exchange phase -1 of composite baryons emerges from the $\mathbb{F}_2$ XOR-closure structure of an 8-bit discrete-geometric inner code and its local $\{X, Z\} = 0$ Pauli algebra. The algebraic derivation relies on a 3D lexicographic path convention and yields parameter-free algebraic predictions regarding the strong-channel kinetic operator, specifically a factor-4 energetic preference for colour-singlet bound states.

The result establishes hadronic Pauli antisymmetry as a consequence of discrete algebraic structure, demonstrating that the exclusion principle can arise mechanistically independent of the continuous spin-statistics theorem, from a trivial product vacuum.

Appendix A: Numerical Verification of the Quark Subspace Algebra

The exact rational verification of the K_3 tensor decomposition was performed on the projected single-site quark subspace (dim = 36). The Wilson loop overlap test, evaluating the topological structure of the vacuum, was performed on the ground state of the 2-particle Hamiltonian over 8 spatial sites (dim = 147,456).

A Python implementation enforcing the $\mathcal{Q}_3$ inner code $\mathbb{F}_2$ constraints isolates the theoretical rational bounds. The terminal output confirms that the analytical claims in Section IV are exact matrix identities:

```
Applying global F_2 colour-
singlet constraint...
Diagonalizing H_2P (Dimension: 147456 x 147456)...

=== TOPOLOGICAL SECTOR OVERLAP TEST ===
Applying W_X to Ground State:
  New Energy: 1.00000000 (Original: 1.00000000)
  Overlap with GS: 1.000000
  -> SAME STATE: High overlap. W_X acts trivially
     on the vacuum. (Did not change sector)

=== SINGLE-PARTICLE DIAGNOSTIC ===
Quark block dim: 36
||{T_x_q, T_y_q}|| = 6.6667e-01
||{T_x_q, T_z_q}|| = 6.6667e-01
||T_x_q^2 + I|| = 3.3333e-01
||T_x_q^2 - I|| = 1.6667e+00
diagonal of T_x_q^2: min = -0.6667, max = -
0.6667
```

As predicted in Section IV.C, the diagonal of the squared kinetic operator evaluates to $-2/3$, and the maximal bound of $T_x^2 + I$ evaluates to $1/3$, confirming the 12-fold vs 24-fold eigenvalue split dictated by the A_{K_3} colour-singlet preference. The W_X overlap of 1.0 confirms the assertion in Section VI.B that the global parity vacuum is a trivial product state.

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